Review of conservation equations

Mass and Momentum
Flux of mass in (kg/s) = $\rho u \cdot dy \cdot dz$

Flux of mass out (kg/s) = $\rho u \cdot dy \cdot dz$

Net Flux of mass in ‘$x$’ = $-\frac{\partial}{\partial x} (\rho u) \cdot dx \cdot dy \cdot dz$

Net Flux of mass in ‘$y$’ = $-\frac{\partial}{\partial y} (\rho v) \cdot dx \cdot dy \cdot dz$

Net Flux of mass in ‘$z$’ = $-\frac{\partial}{\partial z} (\rho w) \cdot dx \cdot dy \cdot dz$
The change of mass per unit time going through the volume element is:

\[
\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \left[ \rho \cdot dx \cdot dy \cdot dz \right]
\]

\[
\frac{\partial}{\partial t} \left[ \rho \cdot dx \cdot dy \cdot dz \right] = \left[ - \frac{\partial}{\partial x} (\rho u) - \frac{\partial}{\partial y} (\rho v) - \frac{\partial}{\partial z} (\rho w) \right] dx \cdot dy \cdot dz
\]

And the change of mass per unit time per unit volume is:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0
\]

which is the same as:

\[
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} + \rho \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] = 0
\]

or

\[
\frac{1}{\rho} \frac{D \rho}{Dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]
This is the Continuity Equation or Equation of Conservation of Mass

How valid is the Boussinesq approximation in the OCEAN?

How would you determine that?

\[
\frac{1}{\rho} \frac{D\rho}{Dt} \approx 0 \quad \text{Boussinesq approximation}
\]

This is the Continuity Equation or Equation of Conservation of Mass

How valid is the Boussinesq approximation in the OCEAN?

How would you determine that?

\[
\frac{\partial u}{\partial x} = \frac{0.1 \text{ m/s}}{100 \text{ km}} = O[10^{-6}]
\]

\[
\frac{D\rho}{Dt} = \frac{1}{\text{sigma-t throughout one day}} = 1 / (24 \times 3600.) = 1.15 \times 10^{-5}
\]

But

\[
\frac{1}{\rho} \frac{D\rho}{Dt} = O[10^{-8}]
\]
Continuity Equation in Bulk Form: \[ R + P_r + V_0 + \theta = E + V_b + \phi \]
Conservation of Salt:

\[ \frac{DS}{Dt} = \text{diffusivity} \]

\[
\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} + v \frac{\partial S}{\partial y} + w \frac{\partial S}{\partial z} = \frac{\partial}{\partial x} \left[ K \frac{\partial S}{\partial x} \right] + \frac{\partial}{\partial y} \left[ K \frac{\partial S}{\partial y} \right] + \frac{\partial}{\partial z} \left[ K \frac{\partial S}{\partial z} \right]
\]

Conservation of Heat:

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \frac{\partial}{\partial x} \left[ \kappa \frac{\partial T}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \kappa \frac{\partial T}{\partial y} \right] + \frac{\partial}{\partial z} \left[ \kappa \frac{\partial T}{\partial z} \right]
\]

Equation of State:

\[ \rho = \rho[S, T, p] \]
Continuity Equation in Bulk Form: \[ R + P_{r} + V_{0} + \theta = E + V_{b} + \phi \]

Salt Conservation Equation in Bulk Form: \[ V_{b}S_{b} = V_{0}S_{0} \]
Conservation of Momentum (Equations of Motion)

\[ m\ddot{a} = \sum \vec{F} \]

\[ \ddot{a} = \sum \frac{\vec{F}}{m} \]

\[ \ddot{a} = \frac{d\vec{V}}{dt} = \begin{bmatrix}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}
\end{bmatrix} \]
\[ \sum \frac{\dot{F}}{m} = \text{Pressure gradient} + \text{friction} + \text{tides} + \text{gravity} + \text{Coriolis} \]

Pressure gradient: Barotropic and Baroclinic
Friction: Surface, bottom, internal
Tides: Boundary condition
Gravity: Only in the vertical
Coriolis: Only in the horizontal

REMEMBER, these are FORCES PER UNIT MASS
\[ \sum \frac{\vec{F}}{m} = \text{Pressure gradient} + \text{friction} + \text{tides} + \text{gravity} + \text{Coriolis} \]

**Pressure gradient**: Barotropic and Baroclinic

**Friction**: Surface, bottom, internal

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REMEMBER, these are FORCES PER UNIT MASS
Hydrostatic Pressure

\[ P_h = -g \int_z^{\eta} \rho \, dz \]

Total Pressure

\[ p = P_a - g \int_z^{\eta} \rho \, dz \]

Pressure gradient force per unit mass

\[-\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{1}{\rho} \left[ \frac{\partial P_a}{\partial x} - \rho g \frac{\partial \eta}{\partial x} - g \int_z^{\eta} \frac{\partial \rho}{\partial x} \, dz \right] \]

Barometric  Barotropic  Baroclinic

Note that even if the density is constant with depth, the horizontal pressure gradient increases with depth if there is a horizontal density gradient.
\[ \sum \frac{\vec{F}}{m} = \text{Pressure gradient} + \text{friction} + \text{tides} + \text{gravity} + \text{Coriolis} \]

Pressure gradient: Barotropic and Baroclinic

Friction: Surface, bottom, internal

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REMEMBER, these are FORCES PER UNIT MASS
Friction

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \left[ A_x \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[ A_y \frac{\partial u}{\partial y} \right] + \frac{\partial}{\partial z} \left[ A_z \frac{\partial u}{\partial z} \right]
\end{bmatrix}
\]

@ surface:
\[
A_z \frac{\partial u}{\partial z} = \frac{\tau_s}{\rho} = \frac{\rho a C_d W_x}{\rho} \left| \vec{W} \right|
\]

@ bottom:
\[
A_z \frac{\partial u}{\partial z} = \frac{\tau_b}{\rho} = \frac{\rho C_b u}{\rho} \left| \vec{V} \right| = C_b u \left| \vec{V} \right| \approx ru; r \approx C_b \left| \vec{V} \right|
\]

@ interior:
\[
A_z \frac{\partial u}{\partial z} = ? \approx f[Ri]; Ri = \frac{-g \frac{\partial \rho}{\rho \frac{\partial u}{\partial z}}}{\frac{\partial u}{\partial z}} + \left[ \frac{\partial v}{\partial z} \right]^2
\]
\[ \sum \frac{\dot{F}}{m} = \text{Pressure gradient} + \text{friction} + \text{tides} + \text{gravity} + \text{Coriolis} \]

Pressure gradient: Barotropic and Baroclinic

Friction: Surface, bottom, internal

Tides: Boundary condition

Gravity: Only in the vertical

Coriolis: Only in the horizontal

REMEMBER, these are FORCES PER UNIT MASS
Gravity

\[ [0, 0, g] = [0, 0, 9.81] \]

Coriolis

\[-fv, fu, 0\]

\[ f = 2\Omega \sin \lambda \]

\[ \Omega = \frac{2\pi}{24h} \]
\[ m\ddot{a} = \sum \vec{F} \]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - fv = -g \frac{\partial \eta}{\partial x} - \frac{g}{\rho} \int_z^\eta \frac{\partial \rho}{\partial x} \, dz + \frac{\partial}{\partial x} \left[ A_x \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[ A_y \frac{\partial u}{\partial y} \right] + \frac{\partial}{\partial z} \left[ A_z \frac{\partial u}{\partial z} \right]
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + fu = -g \frac{\partial \eta}{\partial y} - \frac{g}{\rho} \int_z^\eta \frac{\partial \rho}{\partial y} \, dz + \frac{\partial}{\partial x} \left[ A_x \frac{\partial v}{\partial x} \right] + \frac{\partial}{\partial y} \left[ A_y \frac{\partial v}{\partial y} \right] + \frac{\partial}{\partial z} \left[ A_z \frac{\partial v}{\partial z} \right]
\]

\[
0 = \frac{1}{\rho} \frac{\partial P}{\partial z} + g
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

\[
\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} + v \frac{\partial S}{\partial y} + w \frac{\partial S}{\partial z} = \frac{\partial}{\partial x} \left[ K_x \frac{\partial S}{\partial x} \right] + \frac{\partial}{\partial y} \left[ K_y \frac{\partial S}{\partial y} \right] + \frac{\partial}{\partial z} \left[ K_z \frac{\partial S}{\partial z} \right]
\]

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \frac{\partial}{\partial x} \left[ \kappa_x \frac{\partial T}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \kappa_y \frac{\partial T}{\partial y} \right] + \frac{\partial}{\partial z} \left[ \kappa_z \frac{\partial T}{\partial z} \right]
\]

\[
\rho = \rho[S, T, p]
\]
Classification of Estuaries
What is an estuary?

Estuaries are semi-enclosed coastal bodies of water that experience free communication with the ocean. The ocean salinity is measurably diluted by runoff from the land.

Coastal lagoons are similar to estuaries but typically have a more restricted opening to the ocean, and their salinity is less diluted.

Semi-enclosed coastal body of water free communication with ocean ocean salinity is measurably diluted by runoff
Types of Estuaries According to Their Origin
San Francisco Bay

(d) TECTONIC ESTUARY (Pinet, 2003)
Typical Estuarine Circulation

Time (h) = 1
Types of Estuaries According to Their Stratification (Pinet, 2003)

Competition between tidal forcing and buoyancy forcing

**Factors Affecting Estuaries**

- **High**
  - Minimum
  - Weak
  - Strong

- **Low**
  - Maximum
  - Strong
  - Weak

**Types**

- (a) Salt-Wedge Estuary
  - River
  - Ocean
  - Salinity Profiles
  - Net Circulation

- (b) Partially Mixed Estuary
  - River
  - Ocean
  - Salinity Profiles
  - Net Circulation

- (c) Well-Mixed Estuary
  - River
  - Ocean
  - Salinity Profiles
  - Net Circulation

Competition between tidal forcing and buoyancy forcing
Typical circulation in a fjord
(a) SALT-WEDGE ESTUARY (Pinet, 2003)
Rio de la Plata Estuary
Argentina
Example of fjord
(b) PARTIALLY MIXED ESTUARY (Pinet, 2003)
The mouth of the bay is on the right

Axial Distributions
Chesapeake Bay

Spring of 1999
Wide estuary, showing lateral separation of saltwater and freshwater due to Coriolis deflection (Northern Hemisphere)
Example of Well-Mixed Estuary
Types of Estuaries According to their water balance

SUMMER
S1 > S2; T1 > T2
Inverse Estuary

WINTER
S1 ~ S2; T1 < T2
Inverse Estuary
Looking into lagoon
Red = Water going out
Blue = Water going in
Salt-plug estuary

E ≥ R

M. Tomczak’s Web Site
Classification of Estuaries by Hydrodynamics


Looking at Partially Mixed estuaries and ignoring lateral variability:

momentum balance: \[ \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial}{\partial z} \left[ A_v \frac{\partial u}{\partial z} \right] \]

continuity: \[ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \]

salt balance: \[ u \frac{\partial S}{\partial x} + w \frac{\partial S}{\partial z} = \frac{\partial}{\partial x} \left[ K_x \frac{\partial S}{\partial x} \right] + \frac{\partial}{\partial z} \left[ K_z \frac{\partial S}{\partial z} \right] \]

equation of state: \[ \rho = \rho_r (1 + \beta S) \]

Non-dimensionalizing:
- current velocity as river flow
- salinity as a function of x and z
Results are cast in terms of two dimensionless parameters for the case of zero wind stress:

1) The tidal mixing parameter $M$:

$$M = \frac{K_z K_{x0} B^2}{R^2}; \quad B^2 \text{ is the width of the estuary}$$

2) The gravitational circulation based on the Estuarine Rayleigh number:

$$Ra = \frac{g \beta S_0 H^3}{A_z K_{x0}}$$

We can use surrogates of the above two parameters to characterize the estuary:
A) The circulation parameter, which is the ratio of the net surface current $u_s$ to the mean freshwater velocity through the section $U_f$

$$\frac{u_s}{U_f} = \frac{\text{surface flow speed}}{\text{sectional average speed}}$$

The larger this ratio, the stronger the gravitational circulation. This ratio is typically $\geq 1$

B) The stratification parameter, which is the ratio of the top-to-bottom salinity difference $\partial S$ to the mean salinity over the section $S_0$

$$\frac{\partial S}{S_0} = \frac{S_{\text{surface}} - S_{\text{bottom}}}{\text{sectional average of } S}$$

At mixed conditions, $\partial S = 0$

$$@ \frac{\partial S}{S_0} \geq 1 \Rightarrow \text{strongly stratified}$$
The diffusive fraction of the total upstream salt flux $\nu$ in an estuary can be determined as a function of these two parameters (circulation and stratification)

$$\nu = \frac{K_x \frac{\partial S}{\partial x}}{U_f S_0} = \frac{\text{Diffusive salt flux}}{\text{Total salt flux}}$$

@ $\nu = 1$ gravitational convection ceases; upstream salt flux entirely by diffusion

@ $\nu \to 0$; diffusion is unimportant; upstream salt flux almost entirely by gravitational convection

$0.1 < \nu < 0.9$; both advective and diffusive fluxes are important in the horizontal balance
Note that the advective component of salt flux is not necessarily proportional to salinity stratification.

@ $\nu = 1$ gravitational convection ceases; upstream salt flux entirely by diffusion

@ $\nu \rightarrow 0$; diffusion is unimportant; upstream salt flux almost entirely by gravitational convection

0.1 < $\nu$ < 0.9; both advective and diffusive fluxes are important in the horizontal balance
Types of Estuaries

1) No vertical structure in $u$
   seaward flow at all depths
diffusive flux of salt dominates
   
   1a) well mixed
   1b) stratified

2) Flow reverses with depth
   advective and diffusive fluxes of salt contribute
   
   2a) well mixed
   2b) stratified

3) Strong gravitational circulation - advective flux of salt is dominant

4) Salt wedge
Classification of Estuaries Based on Lateral Structure of Exchange Hydrodynamics
Along-basin:

Pressure Gradient + Friction + Coriolis

\[-fv = -g \frac{\partial \eta}{\partial x} - \frac{g}{\rho} \int_{-H}^{0} \frac{\partial \rho}{\partial x} \, dz + A_z \frac{\partial^2 u}{\partial z^2}\]

Across-basin:

Pressure Gradient + Friction + Coriolis

\[fu = -g \frac{\partial \eta}{\partial y} - \frac{g}{\rho} \int_{-H}^{0} \frac{\partial \rho}{\partial y} \, dz + A_z \frac{\partial^2 v}{\partial z^2}\]

Friction/Coriolis

\[E = A_z / (fH^2)\]
Ekman # -- proxy for dynamical depth

Small $E$

Deep Basin

Large $E$

Shallow Basin
Friction/Coriolis

\[ E = \frac{A_z}{(f H^2)} \]

(red is inflow; white is outflow)
(contours are normalized with the maximum flow; \( A_z \) is constant)

Depth or Width (\( B \))?

\[ Ke = \frac{B}{R} \]

Valle-Levinson et al. (2003, JPO)
Ke = B/R; E = A_z / (f H^2)

(looking into the estuary; orange is inflow; white is outflow)
Highly Frictional \( U_{\text{max}} - U_{\text{min}} \)

Frictionless

Highly Frictional

3131 solutions
Future Challenges